The Waiting Time Distribution for a TDMA Model with a Finite Buffer

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***Abstract*— We obtain detailed analytic formulas for the density and probability distribution of the waiting time in a TDMA model with a finite buffer. On successive intervals of length equal to the duration of a slot, the density is expressed as (infinite) linear combinations of beta densities with positive coefficients. A recursive scheme, obtained by a matrix-analytic derivation, allows for the highly efficient computations of the coefficient sequences.**

I. Introduction

TDMA (Time Division Multiple Access) systems are widely used in various telecommunication applications. Given the wide applicability of TDMA, models for TDMA applications, options, and versions have been extensively studied for over a quarter of a century [1]–[14]. The model here was motivated by the application of GSM paging described in [4]. As will be clear from this paper, this model gives rise to an interesting queueing problem with a certain degree analytical complexity.

The methodological interest of this paper lies in how we handle the complex accounting of the queueing process. A matrix formalism is used to obtain the main structural results. Later, that formalism is unpacked to reveal the analytic form of the density of the waiting time (or delay) distribution. Note that the terms "waiting time" and "delay" are interchangeable in this paper. We shall see that, on each of the successive TDMA fixed intervals, the density is given by an infinite, positive linear combination of beta densities. The coefficients of those linear combinations are computed by a recursive scheme. We so arrive at a nearly explicit characterization of the waiting time density and at an algorithm for its numerical computation.

The remainder of the paper is organized as follows. In Section II we describe the model. Sections III and IV deal with derivations of preliminary quantities. An outline of the mathematical derivation of the delay distribution is given in Section V. The details of that derivation are presented in Sections VI, and VII. Finally in Section VIII we present the computational results for a numerical example, we confirm these by simulation, and we give an intuitive explanation for the behavior of the waiting time density function.

II. The Model

We derive the waiting time distribution of an arbitrary admitted customer to a finite buffer that operates under the following procedure. The customers arrive according to a homogeneous Poisson process of rate λ. If there are fewer than K customers present, an arriving item is *admitted*, otherwise it is lost. The time axis is divided equally into successive frames (slots) of length T. If at the end of a slot, there are j items in the buffer, then with conditional probability $d(i,j)$, i of the j items are removed on a first come first served (FCFS) basis. For $1 \leq j \leq K$, the quantities $\{d(i,j)\}$ satisfy

$$\sum_{i=0}^{j} d(i,j) = 1. \quad (1)$$

That service process has some degree of generality as it can treat cases where message sizes are variable, that is, the larger the messages in the queue, the less are served. It can also apply to the case of demand assigned TDMA whereby the longer the queue in terms of the number of messages, the more messages are served. The latter applies to the case where more TDMA "slots" are allocated to a station where the demand is higher.

An item admitted during the frame $(0,T)$ may be removed at one of the epochs kT, $k \geq 1$. If it is admitted at time $T-u$, $0 \leq u \leq T$, its waiting time is therefore the sum of u and an integer multiple kT of T.

III. The Embedded Markov Chain

Let J_k be the number of items in the buffer at time $kT+$. $\{J_k\}$ is then a Markov chain with state space $\{0,1,\ldots,K\}$.

Its probability transition matrix $P = \{P_{ij}\}$ is given by

$$\begin{aligned} P_{ij} &= P(J_{k+1} = j | J_k = i) \\ &= \sum_{\nu=max(0,j-i)}^{K-i-1} e^{-\lambda T} \frac{(\lambda T)^{\nu}}{\nu!} \; d(i+\nu-j, i+\nu) \\ &\quad + \left[1 - \sum_{\nu=0}^{K-i-1} e^{-\lambda T} \frac{(\lambda T)^{\nu}}{\nu!}\right] d(K-j, K), \end{aligned} \quad (2)$$

for $0 \le i, j \le K$. By $[\pi_0, \pi_1, \ldots, \pi_K]$, we denote the steady-state probabilities of that Markov chain and we assume that these have been computed.

IV. The expected number of items admitted per time frame

Theorem 1: The expected number E^* of packets admitted during a slot of length T is

$$E^* = \sum_{i=0}^{K-1} \pi_i \left[K - i - \sum_{\nu=0}^{K-i-1} e^{-\lambda T} \frac{(\lambda T)^{\nu}}{\nu!} (K-i-\nu)\right]. \quad (3)$$

Proof: Given that there are i, $0 \le i \le K-1$, items in the buffer at the beginning of a slot, the item must arrive so that it can occupy one of the positions r with $i+1 \le r \le K$. Using the indicator random variables of the corresponding events, we readily see that

$$E^* = \sum_{i=0}^{K-1} \pi_i \sum_{r=i+1}^{K} \int_0^T e^{-\lambda u} \frac{(\lambda u)^{r-i-1}}{(r-i-1)!} \; \lambda du, \quad (4)$$

but

$$\int_0^T e^{-\lambda u} \frac{(\lambda u)^{r-i-1}}{(r-i-1)!} \; \lambda du = 1 - \sum_{\nu=0}^{r-i-1} e^{-\lambda T} \frac{(\lambda T)^{\nu}}{\nu!},$$

thus (4) becomes

$$\begin{aligned} E^* &= \sum_{i=0}^{K-1} \pi_i \sum_{r=i+1}^{K} \left[1 - \sum_{\nu=0}^{r-i-1} e^{-\lambda T} \frac{(\lambda T)^{\nu}}{\nu!}\right] \\ &= \sum_{i=0}^{K-1} \pi_i \left[K - i - \sum_{l=0}^{K-i-1} \sum_{\nu=0}^{l} e^{-\lambda T} \frac{(\lambda T)^{\nu}}{\nu!}\right] \\ &= \sum_{i=0}^{K-1} \pi_i \left[K - i - \sum_{\nu=0}^{K-i-1} e^{-\lambda T} \frac{(\lambda T)^{\nu}}{\nu!} (K-i-\nu)\right]. \end{aligned} \quad (5)$$

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The ratio E^*/T is the steady-state rate at which items are admitted to the buffer, so that $(E^*/T)dv$ is the elementary probability of an admission in $(v, v+dv)$.

V. An outline of the derivation

Let $\psi(\cdot)$ be the probability density of the delay of an arbitrary admitted item. In this section, we present an outline of the derivation of $\psi(\cdot)$ with the cumbersome details to be filled in later. We choose the time origin 0 at the beginning of the slot during which the arbitrary item is admitted.

We first derive the expected number $dE^*(u)$ of items admitted during $(0, T)$ whose waiting time lies between u and $u + du$. That derivation is somewhat involved. When that is completed, we note that

$$[dE^*(u)/T]/[E^*/T] = \psi(u)du,$$

is the elementary probability that an arbitrary admitted item waits between u and $u+du$. Therefore, $\psi(\cdot)$ is the probability density of the waiting time distribution.

What requires a well-organized derivation is that the function $E^*(u)$ assumes different analytic forms on the successive intervals $(kT, kT+T)$, $k \ge 0$. To express the first density and to relate the form of the density on a subsequent interval to the preceding one requires somewhat involved bookkeeping. That second task is accomplished by using a convenient matrix formalism.

We must keep track of the buffer content at each epoch $kT+$ and of the position r, $K \ge r \ge 1$, of the item that we are following. While, owing to new arrivals and successive departures, the buffer content can increase and decrease, the position r is non-increasing from one slot to the next. When the tracked item leaves the buffer, we shall say that it reaches position 0.

A. Accounting for the first frame $(0, T)$

It is useful to introduce vectors $\mathbf{g}^*(r'; u)$, for $K \ge r' \ge 1$ and $0 \le u \le T$. The vector $\mathbf{g}^*(r'; u)$ is of dimension $K - r' + 1$. The infinitesimal quantity $g^*_{i'}(r'; u)du$, $K \ge i' \ge r'$, is the expected number of items admitted between $T - u$ and $T - u + du$, evaluated over the event where the buffer content at time $T+$ equals i' and the position of that item at time $T+$ is r'. As there can be at most one such an arrival, $g^*_{i'}(r'; u)du$ is also the elementary probability that between $T - u$ and $T - u + du$ an item is admitted, that its position at time $T+$ is r' and that the buffer content then equals i'. Expressions for the vectors $\mathbf{g}^*(r'; u)$ are derived in Section VI-B.

Next, we define some convenient matrices that serve to account for transactions during the first slot $(0, T)$. The matrices $T_0(r, r'; u)$ are defined for $K \ge r \ge 1$ and for $r' \le r$. The matrix $T_0(r, r'; u)$ is of dimensions $r \times (K - r' + 1)$. Its row indices i run from 0 to $r-1$; its column indices i' from r' to K. The quantity $[T_0(r, r'; u)]_{i,i'}du$ is the elementary conditional probability that, given that the buffer contains i items at time $0+$, an item is admitted into the rth buffer position between $T - u$ and $T - u + du$, and that at time $T+$, there are i' items present and the item we are tracking is now in the position r'.

For $K \ge r \ge 1$, we also define column vectors $\mathbf{T}^0_0(r; u)$ of dimension r. The quantity $[\mathbf{T}^0_0(r; u)]_i du$ is the elementary conditional probability that, given that the buffer content at time $0+$ is i, an item is admitted into the rth buffer position between $T - u$ and $T - u + du$ and departs at time T.

B. Accounting for the subsequent frames $(kT, kT + T)$

To do the accounting for the subsequent slots, we define the $(K - r_1 + 1) \times (K - r_2 + 1)$ matrices $T(r_1, r_2)$, for $K \ge r_1 \ge 1$ and for $r_2 \le r_1$. The row and column indices i_1 and i_2 of $T(r_1, r_2)$ range from K down to r_1 and from K down to r_2 respectively. The element $[T(r_1, r_2)]_{i_1,i_2}$ is the conditional probability that, given that, at the beginning of the slot, there are i_1 items in the buffer with the marked item in position r_1, by the end of the slot, there are i_2 items in the buffer and the item we are tracking has moved to position r_2, with $r_1 \geq r_2 \geq 1$.

For $K \geq r_1 \geq 1$, we also define column vectors $\mathbf{T}^0(r_1)$ of dimension $K - r_1 + 1$. The quantity $[\mathbf{T}^0(r_1)]_{i_1}$ is the conditional probability that, given that, at the beginning of the slot, there are i_1 items in the buffer with the marked item in position r_1, the item we are tracking is removed at the end of that slot.

It will be convenient to consider the matrix T (6, see next page) whose structure is displayed for the representative value $K = 6$.

We see that this matrix is of the form

$$T = \begin{vmatrix} \widetilde{T} & \widetilde{\mathbf{T}}^0 \\ \mathbf{0} & 1 \end{vmatrix}, \tag{7}$$

and we shall later verify that T is stochastic. The derivation of the delay distribution is based on rudiments of the theory of discrete *phase-type* distributions.

The two essential steps of the derivation of that distribution are the following: Let us write $\pi^*(r)$ for the vector $[\pi_{r-1}, \pi_{r-2}, \ldots \pi_0]$. The expected number $dE^*(u)$ of items admitted during $(0, T)$ whose waiting time lies between u and $u + du$ is given by different expressions on the successive intervals $(kT, kT + T), k \geq 0$. Applying the law of total conditional expectation, we see that, for $0 \leq u < T$,

$$dE^*(u) = \sum_{r=1}^{K} \pi^*(r) \mathbf{T}_0^0(r; T - u) du. \tag{8}$$

The direct sum of a finite ordered set of (row)vectors is the vector obtained by concatenating these vectors into a single row vector. For convenience, we form the direct sum $\gamma^*(u)$ of the vectors $\mathbf{g}^*(r; u)$ with r running from K down to 1. Then, a further application of the law of total conditional expectation yields that, on the interval $(kT, kT + T)$, for $k \geq 1$,

$$dE^*(u) = \gamma^*(kT + T - u) \widetilde{T}^{k-1} \widetilde{\mathbf{T}}^0. \tag{9}$$

The method of computation of the function $E^*(u)$ and, therefore, of the probability density of the delay, is now clear in principle. By using equation (8), we evaluate the function on $(0, T)$. Then, recursively forming the vectors $\widetilde{T}^{k-1} \widetilde{\mathbf{T}}^0$, we apply (9) to compute the function for the subsequent frames. However, by getting into the details, the analytic results and the algorithmic procedure can be made much more explicit. These matters are discussed in the next two sections.

VI. The Delay Distribution - The First Slot

The elements of the matrices $T_0(r, r'; u)$, $K \geq r \geq 1$, for $r' \leq r$ are now made explicit. We recall that the quantity $[T_0(r, r'; u)]_{i,i'} du$ is the elementary conditional probability that, given that the buffer content at time $0+$ is i, an item is admitted into the rth buffer position between $T - u$ and $T - u + du$, that at time $T+$, there are i' items in the buffer and the item we are tracking has moved to position r'. Clearly, we can have positive probability only when $r \geq i + 1, r \geq r' \geq 0$. Moreover, the initial state i cannot be K, otherwise no admission during $(0, T)$ is possible. The transition during $(T - u, T)$ can occur with or without the buffer filling up. If it does not, then for $r \geq i + 1, i' \geq r' \geq 1$, $i' + r - r' < K$, there must be $r - r'$ removals at time T. That means that there must be $i' + r - r'$ items just prior to T, so that there are $r - i - 1$ arrivals in $(0, T - u)$ and $i' - r'$ in $(T - u, T)$.

Therefore, for $r \geq i + 1, i' \geq r' \geq 1, i' + r - r' < K$, the quantity $[T_0(r, r'; u)]_{i,i'} du$ is given by

$$\begin{gathered} e^{-\lambda(T-u)} \tfrac{[\lambda(T-u)]^{r-i-1}}{(r-i-1)!} \lambda du \\ \cdot e^{-\lambda u} \tfrac{(\lambda u)^{i'-r'}}{(i'-r')!} d(r - r', i' + r - r'). \end{gathered} \tag{10}$$

If $i' + r - r' = K$, the buffer fills up during $(T - u, T)$. The corresponding expression for that case is

$$\begin{gathered} e^{-\lambda(T-u)} \tfrac{[\lambda(T-u)]^{r-i-1}}{(r-i-1)!} \lambda du \\ \cdot \left[1 - \sum_{\nu=0}^{K-r-1} e^{-\lambda u} \tfrac{(\lambda u)^{\nu}}{\nu!} \right] d(r - r', K), \end{gathered} \tag{11}$$

for $r \geq i + 1, i' = K + r' - r, r' \geq 1$.

The elements of the column vectors $\mathbf{T}_0^0(r; u)$, $K \geq r \geq 1$, are similarly defined. The quantity $[\mathbf{T}_0^0(r; u)]_i du$, the elementary conditional probability that, given that the buffer content at time $0+$ is i, an item is admitted into the rth buffer position between $T - u$ and $T - u + du$ and *departs* at time T, is given by

$$\begin{gathered} e^{-\lambda(T-u)} \tfrac{[\lambda(T-u)]^{r-i-1}}{(r-i-1)!} \lambda du \\ \cdot \left\{ \sum_{j=0}^{K-1-r} e^{-\lambda u} \tfrac{(\lambda u)^j}{j!} \sum_{\nu=r}^{j+r} d(\nu, j + r) \right. \\ \left. + \left[1 - \sum_{j=0}^{K-1-r} e^{-\lambda u} \tfrac{(\lambda u)^j}{j!} \right] \sum_{\nu=r}^{K} d(\nu, K) \right\}, \end{gathered} \tag{12}$$

for $r \geq i + 1$. When $r' = 0$ the item is removed at time T. It then no longer matters how many items remain in the buffer. The marked item is removed at time T if r or more items are removed at that time. The two terms in equation (12) reflect whether or not the buffer fills up in $(T - u, T)$.

The elements of the matrix $T_0(r, r'; u)$ are conveniently expressed as linear combinations of beta densities

$$\beta(y; \alpha, \gamma) = [B(\alpha, \gamma)]^{-1} y^{\alpha-1} (1 - y)^{\gamma-1}, \quad 0 < y < 1, \tag{13}$$

where $B(\alpha, \gamma)$ is the beta function

$$B(\alpha, \gamma) = \int_0^1 v^{\alpha-1} (1 - v)^{\gamma-1} dv.$$

By routine manipulations, we rewrite the elements of $T_0(r, r'; u)$ as

$$\begin{gathered} [T_0(r, r'; u)]_{i,i'} = e^{-\lambda T} \tfrac{(\lambda T)^{r-r'+i'-i}}{(r-r'+i'-i)!} d(r - r', i' + r - r') \\ \cdot \tfrac{1}{T} \beta(1 - \tfrac{u}{T}; r - i; i' - r' + 1), \end{gathered} \tag{14}$$

$$T = \begin{array}{c||cccccc|c} & \underline{6} & \underline{5} & \underline{4} & \underline{3} & \underline{2} & \underline{1} & 0 \\ \hline \underline{6} & T(6,6) & T(6,5) & T(6,4) & T(6,3) & T(6,2) & T(6,1) & \mathbf{T}^0(6) \\ \underline{5} & 0 & T(5,5) & T(5,4) & T(5,3) & T(5,2) & T(5,1) & \mathbf{T}^0(5) \\ \underline{4} & 0 & 0 & T(4,4) & T(4,3) & T(4,2) & T(4,1) & \mathbf{T}^0(4) \\ \underline{3} & 0 & 0 & 0 & T(3,3) & T(3,2) & T(3,1) & \mathbf{T}^0(3) \\ \underline{2} & 0 & 0 & 0 & 0 & T(2,2) & T(2,1) & \mathbf{T}^0(2) \\ \underline{1} & 0 & 0 & 0 & 0 & 0 & T(1,1) & \mathbf{T}^0(1) \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \qquad (6)$$

for $r \geq i+1$, $i' \geq r' \geq 1$, $i'+r-r' < K$, and for $r \geq i+1$, $1 \leq r' \leq r$,

$$\begin{aligned} &[T_0(r,r';u)]_{i,K+r'-r} = \\ &\sum_{j=K-r}^{\infty} e^{-\lambda T} \tfrac{(\lambda T)^{r-i+j}}{(r-i+j)!} d(r-r',K) \\ &\cdot \tfrac{1}{T}\beta(1-\tfrac{u}{T}; r-i; j+1). \end{aligned}$$

Similarly, we rewrite the components of the vectors $\mathbf{T}_0^0(r;u)$. From formula (12) we obtain that for $i \leq r-1$,

$$\begin{aligned} &[\mathbf{T}_0^0(r;u)]_i \\ &= \sum_{j=0}^{K-r-1} e^{-\lambda T} \tfrac{(\lambda T)^{r-i+j}}{(r-i+j)!} \sum_{\nu=r}^{j+r} d(\nu, j+r) \\ &\cdot \tfrac{1}{T}\beta(1-\tfrac{u}{T}; r-i; j+1) \\ &+ \sum_{j=K-r}^{\infty} e^{-\lambda T} \tfrac{(\lambda T)^{r-i+j}}{(r-i+j)!} \sum_{\nu=r}^{K} d(\nu, K) \\ &\cdot \tfrac{1}{T}\beta(1-\tfrac{u}{T}; r-i; j+1). \end{aligned} \qquad (15)$$

A. The density on $(0,T)$

The probability density $\psi(\cdot)$ on the interval $(0,T)$ can now be written explicitly in terms of beta densities. By virtue of formula (8), the density $\psi(\cdot)$ is given by

$$\psi(u) = (E^*)^{-1} \sum_{r=1}^{K} \sum_{i=0}^{r-1} \pi_i [\mathbf{T}_0^0(r;u)]_i. \qquad (16)$$

We set $i = r-h-1$ and carefully interchange the summations in the resulting formulas. That successively yields the expressions (17, see next page) for $0 \leq u \leq T$.

We see that $\psi(u)$ is expressed as an (infinite) positive linear combination of beta densities on $(0,1)$. Correspondingly, the distribution of the delay is given by the same positive linear combination of beta distributions. By calling the incomplete beta function, both can be evaluated by essentially the same algorithm.

Moreover, $\psi(u)$ is of the form

$$\begin{aligned} \psi(u) = &\sum_{h=0}^{\infty} \sum_{j=0}^{\infty} c(h,j)(E^*)^{-1} e^{-\lambda T} \tfrac{(\lambda T)^{h+j+1}}{(h+j+1)!} \\ &\cdot \tfrac{1}{T}\beta(1-\tfrac{u}{T}; h+1; j+1), \end{aligned} \qquad (18)$$

where the coefficient $c(h,j)$ are given by

$$c(h,j) = 0, \quad \text{for} \quad h \geq K, \; j \geq 0,$$

$$c(K-1,j) = \pi_0 d(K,K), \quad \text{for} \quad j \geq 0,$$

and for $0 \leq h \leq K-2$,

$$\begin{aligned} c(h,j) = &\sum_{r=h+1}^{K-j-1} \pi_{r-1-h} \sum_{\nu=r}^{j+r} d(\nu, j+r) \\ &+ \sum_{r=K-j}^{K} \pi_{r-1-h} \sum_{\nu=r}^{K} d(\nu,K), \quad \text{for} \quad 0 \leq j \leq K-h-2, \\ c(h,j) = &\sum_{r=h+1}^{K} \pi_{r-1-h} \sum_{\nu=r}^{K} d(\nu,K), \\ &\text{for} \quad j \geq K-h-1. \end{aligned}$$

B. The auxiliary vectors $\mathbf{g}^*(r';u)$

Next, we give explicit expressions for the vectors $\mathbf{g}^*(r';u)$, for $K \geq r' \geq 1$, for $0 \leq u \leq T$. We recall that $\pi^*(r) = [\pi_{r-1}, \pi_{r-2}, \ldots \pi_0]$. If we pre-multiply the matrix $T_0(r,r';u)$ by $\pi^*(r)$, we obtain a row vector of dimension $K - r' + 1$ and with component indices running from K down to r'. The explicit computation of the component with index $K - r + r'$ requires the second formula in (14); that of all other components utilizes the first formula.

As $g^*_{i'}(r';u)du$ is the elementary probability that between $T-u$ and $T-u+du$ an item is admitted, that its position at time $T+$ is r', and that the buffer content then equals i', we see that

$$g^*_{i'}(r';u) = \sum_{r=r'}^{K} [\pi^*(r) T_0(r,r';u)]_{i'}, \qquad (19)$$

for i' running from K down to r'. We now do a careful accounting of the terms that contribute to each component of $\mathbf{g}^*(r';u)$ and we find that

$$g^*_K(r';u) = \sum_{i=0}^{r'-1} \pi_i [T_0(r',r';u)]^*_{i,K}, \qquad (20)$$

and, for $i' = K-1, \ldots, r'$,

$$\begin{aligned} g^*_{i'}(r';u) = &[\pi^*(K+r'-i') T_0(K+r'-i', r'; u)]^*_{i'} \\ &+ \sum_{r=r'}^{K+r'-i'-1} [\pi^*(r) T_0(r,r';u)]_{i'}, \end{aligned} \qquad (21)$$

where the asterisks remind us of which terms are given by the second, rather than the first formula in (14). We now do those substitutions and we simplify the resulting formulas. It is convenient to evaluate the vectors and their components with decreasing indices, so in the analytic expressions that follow, we define the indices accordingly in (22, see next page).

$$\psi(u) = (E^*)^{-1}\sum_{r=1}^{K-1}\sum_{h=0}^{r-1}\sum_{j=0}^{K-r-1}\pi_{r-1-h}e^{-\lambda T}\frac{(\lambda T)^{h+j+1}}{(h+j+1)!}\sum_{\nu=r}^{j+r}d(\nu,j+r)\frac{1}{T}\beta(1-\frac{u}{T};h+1;j+1)$$

$$+(E^*)^{-1}\sum_{r=1}^{K}\sum_{h=0}^{r-1}\sum_{j=K-r}^{\infty}\pi_{r-1-h}e^{-\lambda T}\frac{(\lambda T)^{h+j+1}}{(h+j+1)!}\sum_{\nu=r}^{K}d(\nu,K)\frac{1}{T}\beta(1-\frac{u}{T};h+1;j+1)$$

$$= (E^*)^{-1}\sum_{h=0}^{K-2}\sum_{j=0}^{K-h-2}e^{-\lambda T}\frac{(\lambda T)^{h+j+1}}{(h+j+1)!}\frac{1}{T}\beta(1-\frac{u}{T};h+1;j+1)\sum_{r=h+1}^{K-j-1}\pi_{r-1-h}\sum_{\nu=r}^{j+r}d(\nu,j+r) \quad (17)$$

$$+(E^*)^{-1}\sum_{h=0}^{K-1}\sum_{j=0}^{\infty}e^{-\lambda T}\frac{(\lambda T)^{h+j+1}}{(h+j+1)!}\frac{1}{T}\beta(1-\frac{u}{T};h+1;j+1)\sum_{r=max(h+1,K-j)}^{K}\pi_{r-1-h}\sum_{\nu=r}^{K}d(\nu,K).$$

$$g^*_K(r';u) = \sum_{h=0}^{r'-1}\sum_{j=K-r'}^{\infty}\pi_{r'-h-1}d(0,K)e^{-\lambda T}\tfrac{(\lambda T)^{h+j+1}}{(h+j+1)!}\tfrac{1}{T}\beta(1-\tfrac{u}{T};h+1;j+1), \quad \text{for} \quad 1 \le r' \le K.$$

$$g^*_{K-v}(r';u) = \sum_{h=0}^{r'+v-1}\pi_{r'+v-h-1}d(v,K)\sum_{j=K-v-r'}^{\infty}e^{-\lambda T}\tfrac{(\lambda T)^{h+j+1}}{(h+j+1)!}\tfrac{1}{T}\beta(1-\tfrac{u}{T};h+1;j+1)$$
$$+\sum_{i=0}^{r'+v-2}\sum_{r=max(r',i+1)}^{r'+v-1}\pi_i e^{-\lambda T}\tfrac{(\lambda T)^{r-i+K-v-r'}}{(r-i+K-v-r')!}d(r-r',K-v+r-r')\tfrac{1}{T}\beta(1-\tfrac{u}{T};r-i;K-v-r'+1), \quad (22)$$
for $1 \le v \le K-r'$.

$$g^*_{K-v}(r';u) = \sum_{h=0}^{r'+v-1}\pi_{r'+v-h-1}d(v,K)\sum_{j=K-v-r'}^{\infty}e^{-\lambda T}\tfrac{(\lambda T)^{h+j+1}}{(h+j+1)!}\tfrac{1}{T}\beta(1-\tfrac{u}{T};h+1;j+1)$$
$$+\left\{\sum_{h=0}^{r'-1}\sum_{i=r'-h-1}^{r'+v-h-2}+\sum_{h=r'}^{r'+v-2}\sum_{i=0}^{r'+v-h-2}\right\}\pi_i e^{-\lambda T}\tfrac{(\lambda T)^{h+K-v-r'+1}}{(h+K-v-r'+1)!}d(i+h-r'+1,K-v+i+h-r'+1) \quad (23)$$
$$\cdot\tfrac{1}{T}\beta(1-\tfrac{u}{T};h+1;K-v-r'+1).$$

In this last sum, we make the change of indices $r-i-1=h$. The result is most conveniently written as the sum of two terms, as follows:

$$\left\{\sum_{h=0}^{r'-1}\sum_{i=r'-h-1}^{r'+v-h-2}+\sum_{h=r'}^{r'+v-2}\sum_{i=0}^{r'+v-h-2}\right\}\pi_i e^{-\lambda T}$$
$$\cdot\tfrac{(\lambda T)^{h+K-v-r'+1}}{(h+K-v-r'+1)!}d(i+h-r'+1,K-v+i+h-r'+1)$$
$$\cdot\tfrac{1}{T}\beta(1-\tfrac{u}{T};h+1;K-v-r'+1).$$

So, finally for $1 \le v \le K-r'$, $g^*_{K-v}(r';u)$ is given by (23).

We see that, for $1 \le r' \le K$, the vector $\mathbf{g}^*(r';u)$ of dimension $K-r'+1$, is of the form

$$\mathbf{g}^*(r';u) = \sum_{h=0}^{\infty}\sum_{j=0}^{\infty}\mathbf{c}^*(r';h,j)e^{-\lambda T}$$
$$\cdot\tfrac{(\lambda T)^{h+j+1}}{(h+j+1)!}\tfrac{1}{T}\beta(1-\tfrac{u}{T};h+1;j+1), \quad (24)$$

where the coefficient vectors $\mathbf{c}^*(r';h,j)$ are given by

$c^*_K(r';h,j) = \pi_{r'-h-1}d(0,K)$, for $0 \le h \le r'-1$, $j \ge K-r'$,
$c^*_K(r';h,j) = 0$, for $h \ge r'$, or for $j \le K-r'-1$,
and for $1 \le v \le K-r'$,
$c^*_{K-v}(r';h,j) = \pi_{r'+v-h-1}d(v,K)$,
for $j > K-v-r'$, $0 \le h \le r'+v-2$,
$c^*_{K-v}(r';h,j) = \pi_0 d(v,K)$, for $j \ge K-v-r'$,
$h = r'+v-1$,
$$c^*_{K-v}(r';h,K-v-r') = \pi_{r'+v-h-1}d(v,K) + \sum_{i=r'-h-1}^{v+r'-h-2}\pi_i d(i+h-r'+1,K-v+i+h-r'+1),$$
for $0 \le h \le r'-1$,
$$c^*_{K-v}(r';h,K-v-r') = \pi_{r'+v-h-1}d(v,K) + \sum_{i=0}^{v+r'-h-2}\pi_i d(i+h-r'+1,K-v+i+h-r'+1),$$
for $r' \le h \le r'+v-2$,
$c^*_{K-v}(r';h,j) = 0$, for $j < K-v-r'$,
or for $h \ge r'+v$. (25)

These coefficient vectors are computed once and stored for use in the recursive computation of the density $\psi(\cdot)$ on the subsequent slots.

VII. The Delay Distribution - The Subsequent Slots

The accounting of the transactions in the buffer content and in the position of the marked item during the subsequent slots is carried out by means of the partitioned matrix T defined in (6). We recall that the elements of T are the conditional probabilities that, given that at the start of the slot there are i_1 items with the marked item in position r_1, by the end of the slot there are i_2 items in the buffer and the marked item has moved to position r_2, with $r_1 \geq r_2 \geq 1$.

The element $[T(r_1, r_2)]_{i_1,i_2}$ is given by

$$\begin{aligned}&[T(r_1, r_2)]_{i_1,i_2} \\ &= e^{-\lambda T} \tfrac{(\lambda T)^{i_2+r_1-r_2-i_1}}{(i_2+r_1-r_2-i_1)!} d(r_1 - r_2, i_2 + r_1 - r_2),\end{aligned} \tag{26}$$

for $i_2 \geq i_1 - r_1 + r_2, i_2 < K - r_1 + r_2$. That is the case where, during the slot, the buffer does not fill up. For future reference, let us call that the form P_C.

The element $[T(r_1, r_2)]_{i_1,K-r_1+r_2}$ is given by

$$\begin{aligned}&[T(r_1, r_2)]_{i_1,K-r_1+r_2} \\ &= \left[1 - \sum_{\nu=0}^{K-1-i_1} e^{-\lambda T} \tfrac{(\lambda T)^\nu}{\nu!}\right] d(r_1 - r_2, K).\end{aligned} \tag{27}$$

It corresponds to $i_2 = K - r_1 + r_2$ and to the buffer filling up during the slot. We call that the form P_D.

For $1 \leq i_1 \leq K$, the components of the vector $\mathbf{T}^0(r_1)$ are given by

$$\begin{aligned}&\sum_{j=0}^{K-1-i_1} e^{-\lambda T} \tfrac{(\lambda T)^j}{j!} \sum_{\nu=r_1}^{i_1+j} d(\nu, i_1 + j) \\ &+ \left[1 - \sum_{\nu=0}^{K-1-i_1} e^{-\lambda T} \tfrac{(\lambda T)^\nu}{\nu!}\right] \sum_{l=r_1}^{K} d(l, K).\end{aligned} \tag{28}$$

The matrices $T(r_1, r_2)$ have further special structure that we display for the representative values $K = 6$, $r_1 = 4$, $r_2 = 4, 3, 2, 1$. The symbols P_C or P_D indicate which of the formulas (26) or (27) that is to be used for the specific indices.

$T(4,4) =$

$i_1 \backslash i_2$	6	5	4
6	P_D	0	0
5	P_D	P_C	0
4	P_D	P_C	P_C

$T(4,3) =$

$i_1 \backslash i_2$	6	5	4	3
6	0	P_D	0	0
5	0	P_D	P_C	0
4	0	P_D	P_C	P_C

$T(4,2) =$

$i_1 \backslash i_2$	6	5	4	3	2
6	0	0	P_D	0	0
5	0	0	P_D	P_C	0
4	0	0	P_D	P_C	P_C

$T(4,1) =$

$i_1 \backslash i_2$	6	5	4	3	2	1
6	0	0	0	P_D	0	0
5	0	0	0	P_D	P_C	0
4	0	0	0	P_D	P_C	P_C

Lemma 1: T is a stochastic matrix.

Proof: For $K \geq i_1 \geq r_1$, the components of the row sum vector $T(r_1, r_2)\mathbf{e}$ are given by

$$\begin{aligned}&[T(r_1, r_2)\mathbf{e}]_{i_1} = \sum_{i_2=i_1-r_1+r_2}^{K-r_1+r_2-1} e^{-\lambda T} \tfrac{(\lambda T)^{i_2+r_1-r_2-i_1}}{(i_2+r_1-r_2-i_1)!} \\ &\cdot d(r_1 - r_2, i_2 + r_1 - r_2) \\ &+ \sum_{j=K-i_1}^{\infty} e^{-\lambda T} \tfrac{(\lambda T)^j}{j!} d(r_1 - r_2, K) \\ &= \sum_{j=0}^{K-i_1-1} e^{-\lambda T} \tfrac{(\lambda T)^j}{j!} d(r_1 - r_2, j + i_1) \\ &+ \sum_{j=K-i_1}^{\infty} e^{-\lambda T} \tfrac{(\lambda T)^j}{j!} d(r_1 - r_2, K).\end{aligned}$$

Now we sum these quantities over r_2 from one to r_1. Finally, we add the term in (28) and, by (1) we obtain an expression that is clearly equal to one. ■

To complete the argument, we verify that $\psi(\cdot)$ is a valid probability density. We integrate the expressions in (8) and (9) and sum over k and check that we so obtain the quantity E^*. We specifically have that

$$\begin{aligned}&\sum_{k=0}^{\infty} \int_{kT}^{kT+T} \psi(u)du = \sum_{k=0}^{\infty} \int_0^T \psi(u + kT)du \\ &= [E^*]^{-1} \left[\sum_{r=1}^{K} \pi^*(r) \int_0^T \mathbf{T}_0^0(r; T - u)du \right. \\ &\left. + \sum_{k=1}^{\infty} \int_0^T \gamma^*(kT + T - u)du \; \widetilde{T}^{k-1} \widetilde{\mathbf{T}}^0 \right].\end{aligned}$$

However, since T is stochastic, it follows that

$$(I - \widetilde{T})^{-1} \widetilde{\mathbf{T}}^0 = \mathbf{e}.$$

It remains to show that

$$\begin{aligned}&[E^*]^{-1} \int_0^T \left[\sum_{r=1}^{K} \pi^*(r) \mathbf{T}_0^0(r; T - u) + \gamma^*(T - u)\mathbf{e}\right] du \\ &= 1.\end{aligned}$$

That can be easily shown without using the fairly involved expressions for $\gamma^*(T-u)$. From the definition of the matrices $T_0(r, r'; u)$ and the vectors $\mathbf{T}_0^0(r; u)$, we readily see that, for $0 \leq i \leq r-1$,

$$\begin{aligned}&\textstyle\sum_{r'=1}^{r} \sum_{i'=r'}^{K} [T_0(r, r'; T - u)]_{i,i'} + [\mathbf{T}_0^0(r; T - u)]_i \\ &= e^{-\lambda u} \tfrac{(\lambda u)^{r-i-1}}{(r-i-1)!} \lambda.\end{aligned}$$

These quantities need to be multiplied by π_i, summed over i with $0 \leq i \leq r-1$, and then summed over r with $1 \leq r \leq K$. We remember that for $0 \leq i \leq r - 1$,

$$\int_0^T e^{-\lambda u} \tfrac{(\lambda u)^{r-i-1}}{(r-i-1)!} \lambda du = 1 - \sum_{\nu=0}^{r-i-1} e^{-\lambda T} \tfrac{(\lambda T)^\nu}{\nu!},$$

so that

$$\begin{aligned}
&[E^*]^{-1}\sum_{r=1}^{K}\sum_{i=0}^{r-1}\pi_i\left[1-\sum_{\nu=0}^{r-i-1}e^{-\lambda T}\frac{(\lambda T)^\nu}{\nu!}\right]\\
&=[E^*]^{-1}\sum_{i=0}^{K-1}\pi_i\sum_{h=0}^{K-1-i}\left[1-\sum_{\nu=0}^{h}e^{-\lambda T}\frac{(\lambda T)^\nu}{\nu!}\right]\\
&=[E^*]^{-1}\sum_{i=0}^{K-1}\pi_i\left[K-i-\sum_{h=0}^{K-1-i}\sum_{\nu=0}^{h}e^{-\lambda T}\frac{(\lambda T)^\nu}{\nu!}\right]\\
&=[E^*]^{-1}\sum_{i=0}^{K-1}\pi_i\left[K-i-\sum_{\nu=0}^{K-1-i}(K-i-\nu)e^{-\lambda T}\frac{(\lambda T)^\nu}{\nu!}\right]\\
&=1.
\end{aligned}$$

A. The final recursive algorithm

To describe the final recursive algorithm concisely, it is convenient to partition the vector $\widetilde{T}^{k-1}\widetilde{\mathbf{T}}^0=\mathbf{w}^*(k)$, $k\geq 1$, into vectors $\mathbf{w}(r',k)$, of dimension $K-r'+1$, for $K\geq r'\geq 1$. The recursive computation of the vectors $\mathbf{w}^*(k)$ is obvious.

We then readily see that, on the interval $(kT,kT+T)$, the density $\psi(\cdot)$ is given by

$$\begin{aligned}
\psi(u)&=[E^*]^{-1}\sum_{r'=1}^{K}\mathbf{g}^*(r';kT+T-u)\mathbf{w}(r',k)\\
&=\sum_{h=0}^{\infty}\sum_{j=0}^{\infty}\{[E^*]^{-1}\sum_{r'=1}^{K}\mathbf{c}^*(r';h,j)\mathbf{w}(r',k)\}\\
&\cdot e^{-\lambda T}\frac{(\lambda T)^{h+j+1}}{(h+j+1)!}\frac{1}{T}\beta(k+1-\tfrac{u}{T};h+1;j+1),
\end{aligned}\tag{29}$$

which is again a positive linear combination of beta densities on $(0,1)$. Its coefficients are given by the term inside the braces. As the coefficients in (24) have been computed, those in (29) are readily evaluated for successive k. Once those coefficients are stored, the density is readily computed at a set of equidistant points in $(kT,kT+T)$ and can be plotted.

Upon integration in (18) and (29), we find that the values $F(kT)$ at the points kT of the probability distribution of the delay are given by

$$F(kT)=\sum_{\nu=1}^{k}\phi_\nu,$$

where

$$\begin{aligned}
\phi_1&=(E^*)^{-1}\sum_{h=0}^{K-2}\sum_{j=0}^{K-h-2}e^{-\lambda T}\frac{(\lambda T)^{h+j+1}}{(h+j+1)!}\\
&\cdot\sum_{r=h+1}^{K-j-1}\pi_{r-1-h}\sum_{\nu=r}^{j+r}d(\nu,j+r)\\
&+(E^*)^{-1}\sum_{h=0}^{K-1}\sum_{j=0}^{\infty}e^{-\lambda T}\frac{(\lambda T)^{h+j+1}}{(h+j+1)!}\\
&\cdot\sum_{r=max(h+1,K-j)}^{K-1}\pi_{r-1-h}\sum_{\nu=r}^{K}d(\nu,K),
\end{aligned}$$

and for $\nu\geq 2$,

$$\begin{aligned}
\phi_\nu&=\sum_{h=0}^{\infty}\sum_{j=0}^{\infty}\{[E^*]^{-1}\sum_{r'=1}^{K}\mathbf{c}^*(r';h,j)\mathbf{w}(r',\nu-1)\}\\
&\cdot e^{-\lambda T}\frac{(\lambda T)^{h+j+1}}{(h+j+1)!}.
\end{aligned}$$

The distribution $F(x)$ at x with $kT<x<kT+T$ is given by

$$F(x)=F(kT)+\int_{kT}^{x}\psi(u)du,\tag{30}$$

and can be evaluated along with the density $\psi(\cdot)$ on that interval simply by substituting the corresponding beta distributions for the beta densities.

We note that, for $\nu\geq 2$, ϕ_ν is the probability that the marked item departs at the end of the (ν -1)st after the interval in which it arrives. ϕ_1 is the probability that it departs at the earliest possible time T. There are no simple analytic expressions for the mean and variance of the delay, but these quantities are simple byproducts of the computation of the distribution $F(\cdot)$.

VIII. Numerical examples

After routine numerical computations of the steady-state probability vector $[\pi_0,\pi_1,\ldots,\pi_K]$ and of the quantity E^*, we implement the recursive scheme for the computation of the coefficient sequences in formulas (18) and (29). For a given value of λT, the Poisson probabilities are computed only once. They are evaluated, starting from the largest term, by applying the standard recurrence relation until terms on either side of the mode become zero or negligible. The remaining factors of the terms in (18) and (29) are computed only when the Poisson factor is non-negligible.

The recurrence is initiated by the coefficients for the first slot $(0,T)$. The density $\psi(u)$ on that interval is evaluated by implementing (17). For the subsequent slots, we perform a matrix multiplication to form the required vector $\mathbf{w}^*(k)=\widetilde{T}^{k-1}\widetilde{\mathbf{T}}^0$, and we evaluate the required coefficient series. From these, the density $\psi(u)$ is readily computed on each interval. Inside the recursive loop, we also compute the probability distribution $F(\cdot)$ by using formula (30) and calling a library routine for the incomplete beta ratio. Getting $F(\cdot)$ along with the density only requires a modicum of additional computation.

Our simulations were initiated by variates from the probability vector $[\pi_0,\pi_1,\ldots,\pi_K]$. Each run consisted of one million time slots from which a histogram and the empirical distribution function of the delay were estimated. For the histogram, the class width was set to one tenth of the slot length T.

In our numerical example, one selected from among many, we consider a buffer of size $K=70$, respectively under light, medium, and high loads ($\lambda T=1,\ 60,\ 600$). For the sake of our example, the parameters $d(i,j)$ were specified as follows: For j even, we formed the $j+1$ integers $1,\ 2,\cdots,j/2+1,j/2,\cdots,2,\ 1$, divided by their sum and identified $d(i,j)$, for $0\leq i\leq j$, with the corresponding ratio. For j odd, we wrote the $j+1$ integers $1,\ 2,\cdots,(j+1)/2,(j+1)/2,\cdots,2,\ 1$,

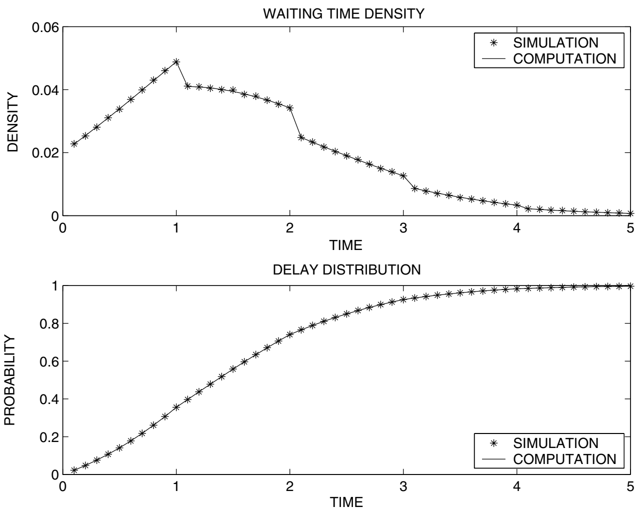

Fig. 1. Computation and simulation results for the waiting time density and delay distribution of the TDMA model, $K = 70$, $\lambda T = 1$.

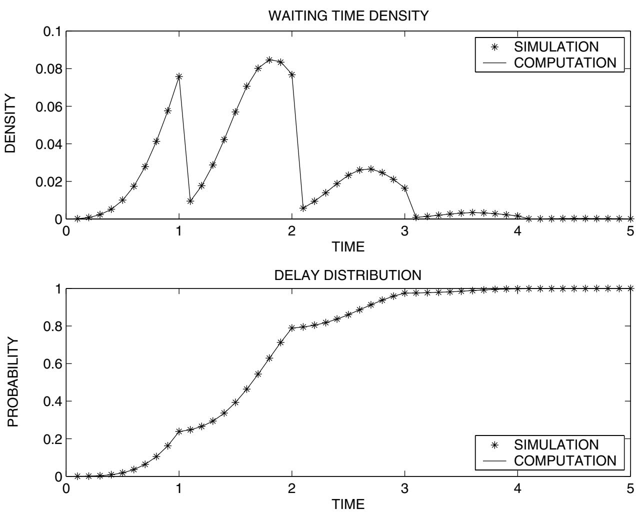

Fig. 2. Computation and simulation results for the waiting time density and delay distribution of the TDMA model, $K = 70$, $\lambda T = 60$.

divided by their sum and similarly identified the $d(i, j)$. That avoids displaying a 71×71 matrix to complete the specification of our example.

Graphs of the corresponding delay densities and distributions, obtained by computation and simulation, are shown in Figures 1-3.

The apparent approach to a discrete density in Figure 3 is to be expected. When the arrival rate is very high, the occasional job that is admitted will arrive very early in the slot and will wait (approximately) for a duration that is a multiple of T.

IX. Conclusions

We have analyzed a TDMA model with a finite buffer, and we have derived exact analytical results which have been confirmed by simulation.

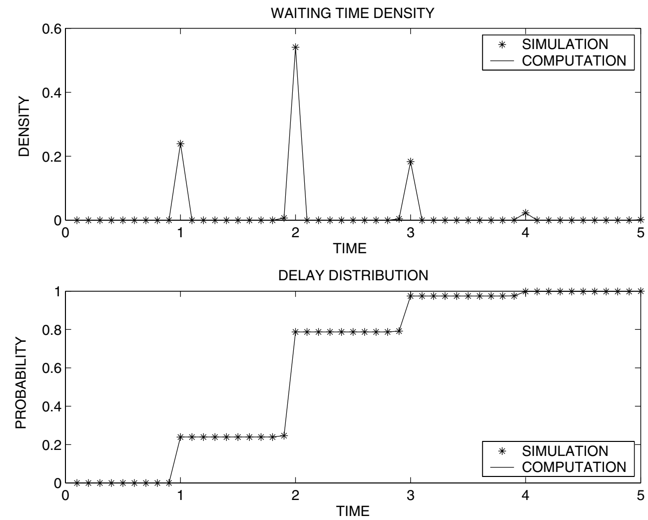

Fig. 3. Computation and simulation results for the waiting time density and delay distribution of the TDMA model, $K = 70$, $\lambda T = 600$.

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